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Characterizations and representations of the Drazin inverse involving idempotents

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ABSTRACT

This paper is to present some results on the Drazin invertibility of products and differences of idempotents. In addition, some formulae for the Drazin inverse of sums, differences and products of idempotents are also established.

0. Introduction

Let X be a complex Banach space. Denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. $\mathcal{R}(T)$ and $\mathcal{N}(T)$ represent the range and the null space of T, respectively. We denote by $\sigma(T)$ the spectrum of T. An element $T \in \mathcal{B}(X)$ is quasi-nilpotent if and only if $\lim_{n\to\infty} \|T^n\|^{1/n} \to 0$. In 1958, Drazin [17] introduced the pseudoinverse for elements of semigroups and polar elements of rings. This notion was generalized by Harte [19] to quasi-polar elements and studied by Koliha [24] in Banach algebras. For $T \in \mathcal{B}(X)$, the (generalized) Drazin inverse of T is the unique (if exists) element $T^d \in \mathcal{B}(X)$ such that

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$$TT^d = T^dT, \quad T^dTT^d = T^d, \quad T - T^2T^d \text{ is quasi-nilpotent.} \tag{1}$$

If there exists an integer k such that $(T - T^2T^d)^k = 0$, then the least integer k is the index of T, denoted by $\mathrm{ind}(T) = k$. Otherwise, we say $\mathrm{ind}(T) = +\infty$. When $\mathrm{ind}(T) = 0$, then the Drazin inverse is reduced into the regular inverse, i.e., $T^d = T^{-1}$. It is well-known that for $T \in \mathcal{B}(X)$, T^d exists if and only if $0 \notin acc[\sigma(T)]$ (for the set of all accumulation points of $\sigma(T)$) and in that case T^d is unique [24]. If T is Drazin invertible, then the spectral idempotent T^π of T corresponding to $\{0\}$ is given by $T^\pi = I - TT^d$. The operator matrix form of T with respect to the space decomposition $X = \mathcal{N}(T^\pi) \oplus \mathcal{R}(T^\pi)$ is given by $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi-nilpotent.

If P and Q are idempotents, then we can consider the matrix representations of P and Q associated with space decompositions $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$. We have

$$P = I \oplus 0 \quad \text{and} \quad Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}. \tag{2}$$

Since $Q^2 = Q$, we obtain that

$$\begin{pmatrix} Q_1^2 + Q_2Q_3 & Q_1Q_2 + Q_2Q_4 \\ Q_3Q_1 + Q_4Q_3 & Q_3Q_2 + Q_4^2 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}. \tag{3}$$

The question of the invertibility of $P - Q$, where P and Q are idempotent operators on a Hilbert space $\mathcal{H}$, is of great interest in operator theory as it is connected with the question of when the space $\mathcal{H}$ is the direct sum $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(Q)$ of the ranges, and with the existence of an idempotent operator F satisfying the equations

$$PF = F, \quad FP = P, \quad Q(I - F) = I - F \quad \text{and} \quad (I - F)Q = Q.$$

These problems were studied by several researchers, for instance, Groß and Trenkler in [18] considered the nonsingularity of $P - Q$ for general matrix projectors; Buckholtz in [1,2], Rakočević in [26], Vidav in [29], Wimmer in [35] discussed the invertibility in the setting of Hilbert spaces; Koliha, Rakočević and Straškraba in [21–24], Rakočević and Wei in [27,28] investigated this question in the setting of C^*-algebra.

In contrast to the above papers, our paper initially considers the more generalized questions of Drazin invertibility and representation formulae of idempotents. The question, which is useful in several applications, such as in the splitting of operators and iterative methods, has been developed by Drazin in his celebrated paper [17, Corollary 1]. Herein, it was proved that

$$(P + Q)^d = P^d + Q^d \ \text{ provided } \ PQ = QP = 0.$$

The general question of how to express $(P + Q)^d$ as a function of P, Q, P^d and Q^d, without any side-conditions, is very difficult and remains open [20]. If P and Q are two idempotents, then we wish to extend Drazin's result to the more general case. In this paper, the formulae for the Drazin inverse of sums, differences and products of idempotents are established. Some of our results recover the results for the regular inverse. We shall assume familiarity with the theory of Drazin inverse as given in [13-17,19,20,24,25,27,28,30-34]. We start by discussing some lemmas.

1. Some lemmas

First we state the following result which was proved in [16] for a bounded linear operator and in [5] for arbitrary elements in a Banach algebra.

Lemma 1.1 (see [5,16]). *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. If A and B are Drazin invertible, then*

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad N = \begin{pmatrix} B & 0 \\ C & A \end{pmatrix}$$

are Drazin invertible and

$$M^d = \begin{pmatrix} A^d & X \\ 0 & B^d \end{pmatrix}, \quad N^d = \begin{pmatrix} B^d & 0 \\ X & A^d \end{pmatrix},$$

where

$$X = (A^d)^2\left[\sum_{n=0}^{\infty}(A^d)^nCB^n\right](I-BB^d) + (I-AA^d)\left[\sum_{n=0}^{\infty}A^nC(B^d)^n\right](B^d)^2 - A^dCB^d.$$

Lemma 1.2 [15]. *Let A and $B \in \mathcal{B}(X)$ be Drazin invertible. If $AB = 0$, then $A + B$ is Drazin invertible and*

$$(A+B)^d = (I-BB^d)\left[\sum_{n=0}^{\infty}B^n(A^d)^n\right]A^d + B^d\left[\sum_{n=0}^{\infty}(B^d)^nA^n\right](I-AA^d).$$

Lemma 1.3 [12, Theorem 2.3]. *AB is Drazin invertible if and only if BA is Drazin invertible and $(AB)^d = A[(BA)^d]^2B$. If A is idempotent, then $A^d = A$. If $AB = BA$, then*

$$(AB)^d = B^dA^d = A^dB^d, \quad A^dB = BA^d \quad \text{and} \quad AB^d = B^dA.$$

Lemma 1.4 [7]. *Let $M \in \mathcal{B}(X \oplus Y)$ have the operator matrix form*

$$M = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \tag{4}$$

Then M is Drazin invertible if and only if AB (or BA) is Drazin invertible. In this case,

$$M^D = \begin{pmatrix} 0 & (AB)^DA \\ B(AB)^D & 0 \end{pmatrix} = \begin{pmatrix} 0 & A(BA)^D \\ (BA)^DB & 0 \end{pmatrix}.$$

2. Drazin invertibility of idempotents

In this section, we present some equivalent conditions for the Drazin invertibility of $P + Q$, $P - Q$, PQ and $PQ \pm QP$, in the case when P and Q are idempotents.

Theorem 2.1. *Let P and Q be idempotents given by Eq. (2). Then*

(1) *Q_1 is Drazin invertible if and only if $I - Q_4$ is Drazin invertible,*
(2) *$I - Q_1$ is Drazin invertible if and only if Q_4 is Drazin invertible.*

Proof. (1) Since

$$(\lambda - I + P)(\lambda - P - Q)(\lambda - I + Q) = \lambda\left[(\lambda-1)^2 - PQ\right],$$

$\sigma(P) \subset \{0, 1\}$ and $\sigma(Q) \subset \{0, 1\}$, we conclude that for $\lambda \in \mathbb{C} \setminus \{0, 1\}$, $\lambda \in \sigma(P + Q)$ if and only if $(\lambda - 1)^2 \in \sigma(PQ)$. Hence, $0 \notin acc[\sigma(1 - P - Q)]$ if and only if $0 \notin acc[\sigma(PQ)]$, and $0 \notin acc[\sigma(P + Q)]$ if and only if $0 \notin acc[\sigma(I - PQ)]$. From Eqs. (2) and (3) we can derive that

$$\sigma\left[(I-P-Q)^2\right] = \sigma[Q_1 \oplus (I-Q_4)] = \sigma(Q_1) \cup \sigma(I-Q_4)$$

and $\sigma(PQ) = \sigma(Q_1) \cup \{0\}$. This shows that $0 \notin acc[\sigma(Q_1)]$ if and only if $0 \notin acc[\sigma(I - Q_4)]$, i.e., Q_1 is Drazin invertible if and only if $I - Q_4$ is Drazin invertible.

(2) Similarly, we have

$$(\lambda - I + P)(\lambda - P + Q)(\lambda + I - Q) = \lambda(\lambda^2 - I + PQ)$$

and $(P - Q)^2 = (I - Q_1) \oplus Q_4$. We obtain that for $\lambda \in \mathbb{C} \setminus \{0, 1\}$, $\lambda \in \sigma(P - Q)$ if and only if $\lambda^2 \in \sigma(I - PQ)$. Hence, $0 \notin acc[\sigma(P - Q)]$ if and only if $0 \notin acc[\sigma(I - PQ)]$. Note that

$$\sigma\left[(P-Q)^2\right] = \sigma(I-Q_1) \cup \sigma(Q_4) \quad \text{and} \quad \sigma(I-PQ) = \sigma(I-Q_1) \cup \{0\}.$$

We obtain that $I - Q_1$ is Drazin invertible if and only if Q_4 is Drazin invertible. $\square$

Using the proof of Theorem 2.1, we present some equivalent conditions for the Drazin invertibility of the sum and product.

Theorem 2.2. *Let P and Q be idempotents given by Eq.* (2). *The following statements are equivalent*:

(1) $P - Q$ *is Drazin invertible*,
(2) $P + Q$ *is Drazin invertible*,
(3) $I - PQ$ *is Drazin invertible*,
(4) $I - Q_1$ *is Drazin invertible*.

Proof. From the proof of Theorem 2.1, we know that (1)–(3) are equivalent to the Drazin invertibility of $I - Q_1$. □

It is worth pointing out that in [21, Theorem 2.1], Koliha has proved that $P - Q$ is invertible if and only if $P + Q$ and $I - PQ$ are invertible. Theorem 2.2 shows that $P - Q$ is Drazin invertible if and only if $P + Q$ is Drazin invertible. Similarly, the following results are also derived by Theorem 2.1.

Theorem 2.3. *Let P and Q be idempotents given by Eq.* (2). *The following statements are equivalent*:

(1) PQ *is Drazin invertible*,
(2) $I - P - Q$ *is Drazin invertible*,
(3) Q_1 *is Drazin invertible*.

Theorem 2.4. *Let P and Q be idempotents given by Eq.* (2). *The following statements are equivalent*:

(1) $PQ - QP$ *is Drazin invertible*,
(2) $PQ + QP$ *is Drazin invertible*,
(3) PQ *and* $P - Q$ *are Drazin invertible*,
(4) Q_1 *and* $I - Q_1$ *are Drazin invertible*.

Proof. Since Q satisfies Eq. (3), we have $Q_2Q_3 = Q_1 - Q_1^2$. By Lemma 1.4, we know that $PQ - QP$ is Drazin invertible if and only if Q_2Q_3 is Drazin invertible, which is equivalent to that Q_1 and $I - Q_1$ are Drazin invertible.

Since $PQ + QP = -(P + Q)(I - P - Q) = -(I - P - Q)(P + Q)$, by Lemma 1.3, we get that $PQ + QP$ is Drazin invertible if and only if $P + Q$ and $I - P - Q$ are Drazin invertible. By Theorems 2.2 and 2.3, we have that (2)–(4) are all equivalent. □

3. Representations for the Drazin inverse

Let P and Q be two idempotents. We provide representations for the Drazin inverse of $P - Q$ and $PQ - QP$ under the assumption that P and Q are given by Eq. (2).

Theorem 3.1. *Let P and Q be idempotents given by Eq.* (2).

(1) *If $P - Q$ is Drazin invertible, then* $(I - Q_1)^dQ_2 = Q_2Q_4^d, Q_3(I - Q_1)^d = Q_4^dQ_3$ *and*

$$(P - Q)^d = \begin{pmatrix} (I - Q_1)^d(I - Q_1) & -(I - Q_1)^dQ_2 \\ -Q_4^dQ_3 & -Q_4^dQ_4 \end{pmatrix}. \tag{5}$$

(2) *If PQ is Drazin invertible, then* $Q_1^dQ_2 = Q_2(I - Q_4)^d, Q_3Q_1^d = (I - Q_4)^dQ_3$ *and*

$$(I - P - Q)^d = \begin{pmatrix} -Q_1^dQ_1 & -Q_1^dQ_2 \\ -(I - Q_4)^dQ_3 & (I - Q_4)^d(I - Q_4) \end{pmatrix}. \tag{6}$$

(3) *If $P-Q$ and PQ are Drazin invertible, then*

$$(PQ-QP)^d = \begin{pmatrix} 0 & -Q_1^d(I-Q_1)^dQ_2 \\ Q_3Q_1^d(I-Q_1)^d & 0 \end{pmatrix}. \tag{7}$$

Proof. (1) Let P and Q have the form (2). Since Q satisfies Eq. (3), after simple calculations we obtain $(P-Q)^2=(I-Q_1)\oplus Q_4$. Then

$$\begin{aligned}(P-Q)^d &= \left[(P-Q)^2\right]^d (P-Q) = (P-Q)\left[(P-Q)^2\right]^d \\ &= \begin{pmatrix} I-Q_1 & 0 \\ 0 & Q_4 \end{pmatrix}^d \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix} = \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix}\begin{pmatrix} I-Q_1 & 0 \\ 0 & Q_4 \end{pmatrix}^d \\ &= \begin{pmatrix} (I-Q_1)^d(I-Q_1) & -(I-Q_1)^dQ_2 \\ -Q_4^dQ_3 & -Q_4^dQ_4 \end{pmatrix} = \begin{pmatrix} (I-Q_1)(I-Q_1)^d & -Q_2Q_4^d \\ -Q_3(I-Q_1)^d & -Q_4Q_4^d \end{pmatrix}.\end{aligned}$$

Hence $(I-Q_1)^dQ_2=Q_2Q_4^d, Q_3(I-Q_1)^d=Q_4^dQ_3$ and Eq. (5) holds.

(2) Note that $(I-P-Q)^2=Q_1\oplus(I-Q_4)$. Similar to the proof of item (1), we can obtain item (2).

(3) Since $PQ-QP=\begin{pmatrix} 0 & Q_2 \\ -Q_3 & 0 \end{pmatrix}$, the result can be obtained directly by Lemma 1.4. □

According to Eq. (3) and Theorem 3.1, we can get the following results easily.

Theorem 3.2. *Let P and Q be idempotents given by Eq. (2). If $P-Q$ is Drazin invertible, then*

$$\begin{aligned}&(P-Q)^\pi = (I-Q_1)^\pi \oplus Q_4^\pi, \\ &(I-P-Q)^\pi = Q_1^\pi \oplus (I-Q_4)^\pi, \\ &(PQ-QP)^\pi = (Q_1^2-Q_1)^\pi \oplus (Q_4-Q_4^2)^\pi.\end{aligned} \tag{8}$$

4. Some formulae for the Drazin inverse of the idempotents

To give explicit formulae for $(P-Q)^d$ and $(P+Q)^d$, we define

$$F=P(P-Q)^d, \quad G=(P-Q)^dP \quad \text{and} \quad H=(P-Q)^d(P-Q). \tag{9}$$

Now, we are able to prove the following result.

Theorem 4.1. *Let P and Q be idempotents, F, G and H given by Eq. (9). If $P-Q$ is Drazin invertible, then F, G and H are idempotents and*

(1) $F=(P-Q)^d(I-Q), \mathcal{R}(F)=\mathcal{R}\left([I-(P-Q)^\pi]P\right)$,
(2) $G=(I-Q)(P-Q)^d, \mathcal{N}(G)=\mathcal{N}(P)\oplus\mathcal{R}\left[(P-Q)^\pi P\right]$.

Proof. Let P and Q be given by Eq. (2). From Theorem 3.1, it is easy to obtain that

$$F=\begin{pmatrix} (I-Q_1)^d(I-Q_1) & -(I-Q_1)^dQ_2 \\ 0 & 0 \end{pmatrix}, \tag{10}$$

with

$$G=\begin{pmatrix} (I-Q_1)^d(I-Q_1) & 0 \\ -Q_3(I-Q_1)^d & 0 \end{pmatrix}, \tag{11}$$

and $H = (I - Q_1)^d(I - Q_1) \oplus Q_4^d Q_4$. So we have

$$\mathcal{R}(F) = \mathcal{R}\left[(I - Q_1)^d(I - Q_1)\right] = \mathcal{R}\left([I - (P - Q)^\pi]P\right)$$

and

$$\begin{aligned}\mathcal{N}(G) &= \mathcal{N}(P) \oplus \mathcal{N}\left[(I - Q_1)^d(I - Q_1)\right] \\ &= \mathcal{N}(P) \oplus \mathcal{R}\left[(I - Q_1)^\pi\right] \\ &= \mathcal{N}(P) \oplus \mathcal{R}\left[(P - Q)^\pi P\right].\end{aligned}$$

Since item (2) can be proved in the same way, as a sample we only prove that $F^2 = F$ and $F = P(P - Q)^d = (P - Q)^d(I - Q)$. By Eq. (3) and Theorem 3.1, we have

$$\begin{aligned}(P - Q)^d(I - Q) &= \begin{pmatrix} (I - Q_1)^d(I - Q_1) & -(I - Q_1)^d Q_2 \\ -Q_4^d Q_3 & -Q_4^d Q_4 \end{pmatrix}\begin{pmatrix} I - Q_1 & -Q_2 \\ -Q_3 & I - Q_4 \end{pmatrix} \\ &= \begin{pmatrix} (I - Q_1)^d(I - Q_1) & -(I - Q_1)^d Q_2 \\ 0 & 0 \end{pmatrix} \\ &= P(P - Q)^d = F.\end{aligned}$$

Moreover,

$$\begin{aligned}F^2 &= (P - Q)^d(I - Q)P(P - Q)^d = (P - Q)^d(I - Q)(P - Q)(P - Q)^d \\ &= P(P - Q)^d(P - Q)(P - Q)^d = P(P - Q)^d = F. \qquad \square\end{aligned}$$

By simple algebraic techniques, we can deduce more relations among F, G and H.

$$QH = HQ, \quad G(I - Q) = (I - Q)F \quad \text{and} \quad FP = PG = PH = HP.$$

Subsequently we present two formulae about $(P - Q)^d$ and $(P + Q)^d$, where P and Q are idempotents.

Theorem 4.2. *Let P and Q be idempotents and $(P + Q)(P - Q)^\pi$ be quasi-nilpotent. If F, G and H are given by Eq. (9), then $(P - Q)^\pi = (P + Q)^\pi$ and*

(1) $(P + Q)^d = (P - Q)^d(P + Q)(P - Q)^d$,
(2) $(P - Q)^d = (P + Q)^d(P - Q)(P + Q)^d$,
(3) $(P - Q)^d = F + G - H$,
(4) $(P + Q)^d = (2G - H)(F + G - H)$.

Proof. Denote by $X = (P - Q)^d(P + Q)(P - Q)^d$. By Theorem 4.1, we have

$$\begin{aligned}(P + Q)X &= (P + Q)(P - Q)^d(P + Q)(P - Q)^d \\ &= (P - Q)^d(I - P + I - Q)(P + Q)(P - Q)^d \\ &= (P - Q)^d(P - Q)^2(P - Q)^d \\ &= (P - Q)(P - Q)^d,\end{aligned}$$

with

$$\begin{aligned}X(P + Q) &= (P - Q)^d(P + Q)(P - Q)^d(P + Q) \\ &= (P - Q)^d(P + Q)(I - P + I - Q)(P - Q)^d \\ &= (P - Q)^d(P - Q)^2(P - Q)^d \\ &= (P - Q)(P - Q)^d\end{aligned}$$

and

$$\begin{aligned} X(P+Q)X &= X(P-Q)(P-Q)^d \\ &= (P-Q)^d(P+Q)(P-Q)^d(P-Q)(P-Q)^d \\ &= (P-Q)^d(P+Q)(P-Q)^d \\ &= X. \end{aligned}$$

Since

$$(P+Q)-(P+Q)^2X = (P+Q)[I-(P-Q)(P-Q)^d] = (P+Q)(P-Q)^\pi$$

is quasi-nilpotent, by the definition (1) of the Drazin inverse, we know $X = (P-Q)^d(P+Q)(P-Q)^d = (P+Q)^d$. By the above proof, we can deduce that

$$(P+Q)^\pi = I-(P+Q)(P+Q)^d = I-(P-Q)(P-Q)^d = (P-Q)^\pi.$$

Now,

$$\begin{aligned} &(P+Q)^d(P-Q)(P+Q)^d \\ &= \left[(P-Q)^d(P+Q)(P-Q)^d\right](P-Q)\left[(P-Q)^d(P+Q)(P-Q)^d\right] \\ &= \left[(P-Q)^d(P+Q)(P-Q)^d\right](P+Q)(P-Q)^d \\ &= X(P+Q)(P-Q)^d \\ &= (P-Q)^d(P-Q)(P-Q)^d \\ &= (P-Q)^d. \end{aligned}$$

Moreover, from Eqs. (5), (10) and (11), it is easy to see that

$$(P-Q)^d = F+G-H.$$

Hence, by Theorem 4.1, $(P-Q)^dQ = (P-Q)^d - F = G-H$ and

$$\begin{aligned} (P+Q)^d &= (P-Q)^d(P+Q)(P-Q)^d \\ &= (P-Q)^dP(P-Q)^d + (P-Q)^dQ(P-Q)^d \\ &= G(F+G-H)+(G-H)(F+G-H) \\ &= (2G-H)(F+G-H). \qquad \square \end{aligned}$$

It is convenient to write $P_{S,T}$ for the idempotent with range S and kernel T. If $\operatorname{ind}(P-Q)=0$, then $H=I, F=P_{\mathcal{R}(P),\mathcal{N}(Q)}$ and $G=P_{\mathcal{N}(Q),\mathcal{N}(P)}$ we can provide explicit formulae for the inverses of sum and difference of two idempotents.

Corollary 4.1 [21, Theorem 2.2]. *Let P and Q be idempotents and $P-Q$ be invertible. If F, G and H are given by Eq. (9), then*

(1) $(P+Q)^{-1} = (P-Q)^{-1}(P+Q)(P-Q)^{-1}$,
(2) $(P-Q)^{-1} = (P+Q)^{-1}(P-Q)(P+Q)^{-1}$,
(3) $(P-Q)^{-1} = F+G-I$,
(4) $(P+Q)^{-1} = (2G-I)(F+G-I)$.

Now we present the main results of this section.

Theorem 4.3. *Let P and Q be idempotents. If $P-Q$ is Drazin invertible, then*

$$(P-Q)^d = (I-PQ)^d(P-PQ)+(P+Q-PQ)^d(PQ-Q). \tag{12}$$

Proof. Let P and Q have the form (2). By Lemma 1.1, we have $(I-PQ)^dP=(I-Q_1)^d\oplus 0$ and $(P+Q-PQ)^d(I-P)=0\oplus Q_4^d$. It follows from Theorem 3.1, item (1) that

$$\begin{aligned}(P-Q)^d&=\begin{pmatrix}(I-Q_1)^d(I-Q_1) & -(I-Q_1)^dQ_2\\ -Q_4^dQ_3 & -Q_4^dQ_4\end{pmatrix}\\ &=(I-PQ)^d(P-PQ)+(P+Q-PQ)^d(PQ-Q).\qquad\square\end{aligned}$$

Theorem 4.4. *Let P and Q be idempotents. If $P-Q$ and PQ are Drazin invertible, then*

$$\begin{aligned}&(PQ-QP)^d=(PQP)^d(P-Q)^d-(P-Q)^d(PQP)^d,\\ &(PQ+QP)^d=(P+Q)^d(P+Q-I)^d.\end{aligned}\tag{13}$$

Proof. Let P and Q have the form (2). Since $(PQP)^d=Q_1^d\oplus 0$, by Lemmas 1.4 and 3.1, we have

$$\begin{aligned}(PQ-QP)^d&=\begin{pmatrix}0 & Q_2\\ -Q_3 & 0\end{pmatrix}^d=\begin{pmatrix}0 & -Q_1^d(I-Q_1)^dQ_2\\ Q_3Q_1^d(I-Q_1)^d & 0\end{pmatrix}\\ &=\begin{pmatrix}Q_1^d(I-Q_1)^d(I-Q_1) & -Q_1^d(I-Q_1)^dQ_2\\ 0 & 0\end{pmatrix}\\ &\quad-\begin{pmatrix}Q_1^d(I-Q_1)^d(I-Q_1) & 0\\ -Q_3Q_1^d(I-Q_1)^d & 0\end{pmatrix}\\ &=(PQP)^d(P-Q)^d-(P-Q)^d(PQP)^d.\end{aligned}$$

Since

$$PQ+QP=(P+Q)(P+Q-I)=(P+Q-I)(P+Q),$$

by Lemma 1.3, we have

$$(PQ+QP)^d=(P+Q)^d(P+Q-I)^d.\qquad\square$$

Theorem 4.5. *Let P and Q be idempotents. If PQ is Drazin invertible, then*

$$\begin{aligned}&(PQP)^d=\left[(I-P-Q)^d\right]^2P,\\ &(PQ)^d=\left[(PQP)^d\right]^2Q=\left[(I-P-Q)^d\right]^4PQ.\end{aligned}\tag{14}$$

Proof. Since

$$PQP=(I-P-Q)^2P=P(I-P-Q)^2,$$

by Lemma 1.3, we derive

$$(PQP)^d=\left[(I-P-Q)^d\right]^2P=P\left[(I-P-Q)^d\right]^2$$

and

$$\begin{aligned}(PQ)^d&=(PPQ)^d=P\left[(PQP)^d\right]^2PQ\\ &=\left[(PQP)^d\right]^2Q=\left[(I-P-Q)^d\right]^4PQ.\qquad\square\end{aligned}$$

If $\mathrm{ind}(P-Q)=0$, Koliha and Rakočević had proved that ([21, Theorem 7.5.1])

$$(I-PQP)^{-1}=I-P+P(P-Q)^{-2}.$$

In fact, we can obtain some more general results on the Drazin inverse.

Theorem 4.6. *Let P and Q be idempotents. If $P-Q$ is Drazin invertible, then*

(1) $(P-PQP)^d = P[(P-Q)^d]^2 = [(P-Q)^d]^2P$,
(2) $(I-PQP)^d = I-P+P[(P-Q)^d]^2$,
(3) $(I-PQ)^d = I-P+\left[(P-Q)^d\right]^2[P+PQ(I-P)]+\left[\sum_{n=0}^{\infty}(P-Q)^{\pi}(P-Q)^{2n}\right]PQ(P-I)$.

Proof. (1) Since $P-PQP = P(P-Q)^2 = (P-Q)^2P$, by Lemma 1.3, we have

$$(P-PQP)^d = \left[P(P-Q)^2\right]^d = P\left[(P-Q)^d\right]^2 = \left[(P-Q)^d\right]^2 P.$$

(2) It follows from Lemma 1.2 that

$$\begin{aligned}(I-PQP)^d &= (I-P+P-PQP)^d\\ &= \left[(P-Q)^2P+(I-P)\right]^d\\ &= P\left(P\left[(P-Q)^d\right]^2\right)+(I-P)\left\{I-P(P-Q)^2\left[(P-Q)^d\right]^2P\right\}\\ &= P\left[(P-Q)^d\right]^2+I-P.\end{aligned}$$

(3) Let P and Q have the form (2). By the proof of Theorem 3.1, item (1) and the results of Theorem 3.2, we have

$$\left[(P-Q)^d\right]^2 = (I-Q_1)^d \oplus Q_4^d, \quad (P-Q)^{\pi} = (I-Q_1)^{\pi} \oplus Q_4^{\pi}.$$

It follows that

$$\begin{aligned}&\left[(P-Q)^d\right]^2[P+PQ(I-P)]\\ &= \begin{pmatrix}(I-Q_1)^d & 0\\ 0 & Q_4^d\end{pmatrix}\left[\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & Q_2\\ 0 & 0\end{pmatrix}\right]\\ &= \begin{pmatrix}(I-Q_1)^d & (I-Q_1)^dQ_2\\ 0 & 0\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}&\left[\sum_{n=0}^{\infty}(P-Q)^{\pi}(P-Q)^{2n}\right]PQ(P-I)\\ &= \sum_{n=0}^{\infty}\begin{pmatrix}(I-Q_1)^{\pi} & 0\\ 0 & Q_4^{\pi}\end{pmatrix}\begin{pmatrix}I-Q_1 & 0\\ 0 & Q_4\end{pmatrix}^n\begin{pmatrix}0 & -Q_2\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & \sum_{n=0}^{\infty}(I-Q_1)^{\pi}(I-Q_1)^n(-Q_2)\\ 0 & 0\end{pmatrix}.\end{aligned}$$

By Lemma 1.1, we obtain

$$\begin{aligned}(I-PQ)^d &= \begin{pmatrix}I-Q_1 & -Q_2\\ 0 & I\end{pmatrix}\\ &= \begin{pmatrix}(I-Q_1)^d & \sum_{n=0}^{\infty}(I-Q_1)^{\pi}(I-Q_1)^n(-Q_2)+(I-Q_1)^dQ_2\\ 0 & I\end{pmatrix}\end{aligned}$$

$$= I - P + \left[(P-Q)^d\right]^2 [P + PQ(I-P)]$$
$$+ \left[\sum_{n=0}^{\infty}(P-Q)^{\pi}(P-Q)^{2n}\right] PQ(P-I). \qquad \square$$

If $P - Q$ is Drazin invertible, then we can develop more relations between the Drazin inverses $(P-PQP)^d$, $(P-PQ)^d$ and $(P-QP)^d$.

Theorem 4.7. *Let P and Q be idempotents. If $P - Q$ is Drazin invertible, then we have the following results:*

(1) $(P-PQP)^d = P(I-PQP)^d = (I-PQP)^d P,$

(2) $(P-PQ)^d = [(I-PQP)^d]^2(P-PQ) = [(P-PQP)^d]^2(I-PQ),$

(3) $(P-QP)^d = (P-QP)[(I-PQP)^d]^2 = (I-QP)[(P-PQP)^d]^2,$

(4) $$\begin{aligned}(I-PQ)^d P &= P(P-QP)^d = (P-PQ)^d P\\ &= P(I-PQP)^d = (I-PQP)^d P\\ &= P(I-QP)^d = (P-PQP)^d,\end{aligned}$$

(5) $$\begin{aligned}(I-QP)^d Q &= Q(Q-PQ)^d = (Q-QP)^d Q\\ &= Q(I-QPQ)^d = (I-QPQ)^d Q\\ &= Q(I-PQ)^d = (Q-QPQ)^d.\end{aligned}$$

Proof. Let P and Q have the matrix representations of the form (2). If $P - Q$ is Drazin invertible, then $I - Q_1$ is Drazin invertible.

(1) Since $(P-PQP)^d = (I-Q_1)^d \oplus 0$ and $(I-PQP)^d = (I-Q_1)^d \oplus I$, it follows that $(P-PQP)^d = P(I-PQP)^d = (I-PQP)^d P$.

(2), (3) Since

$$(P-PQ)^d = \begin{pmatrix} (I-Q_1)^d & -[(I-Q_1)^d]^2 Q_2 \\ 0 & 0 \end{pmatrix}$$

and

$$(P-QP)^d = \begin{pmatrix} (I-Q_1)^d & 0 \\ -Q_3[(I-Q_1)^d]^2 & 0 \end{pmatrix}.$$

Then it is easy to derive that

$$\begin{aligned}(P-PQ)^d &= \left[(I-PQP)^d\right]^2 (P-PQ)\\ &= \left[(P-PQP)^d\right]^2 (I-PQ)\\ &= \left[(P-PQP)^d\right]^2 (P-PQ)\end{aligned}$$

and

$$\begin{aligned}(P-QP)^d &= (P-QP)\left[(I-PQP)^d\right]^2\\ &= (I-QP)\left[(P-PQP)^d\right]^2\\ &= (P-QP)\left[(P-PQP)^d\right]^2.\end{aligned}$$

(4) From the proofs of the item (1)–(3), we have

$$
\begin{aligned}
(I-PQ)^dP &= P(P-QP)^d = (P-PQ)^dP \\
&= P(I-PQP)^d = (I-PQP)^dP \\
&= P(I-QP)^d = (P-PQP)^d \\
&= (I-Q_1)^d \oplus 0.
\end{aligned}
$$

(5) Switch the roles of P and Q in item (4). □

5. Some special cases for the Drazin inverse of the idempotents

In this section we derive formulae for the Drazin inverse of the idempotents under some special conditions. We can show that all results from [14] still hold for idempotents on a Banach space. Moreover, by Theorems 4.1–4.7, we obtain the following results immediately.

Corollary 5.1. *Let P and Q be two idempotents. If $PQP = 0$, then*

(1) $(P-PQ)^d = P-PQ$, $(P-QP)^d = P-QP$,
(2) ([14, Theorem 2.1])
$(P-Q)^D = P-Q-QPQ$,
$(P+Q)^D = P+Q-2PQ-2QP+3QPQ$,
(3) $(PQ)^d = 0$, $(PQ-QP)^d = 0$, $[(I-P-Q)^d]^2P = 0$,
(4) $P = (I-PQ)^dP = P(I-QP)^d = P[(P-Q)^d]^2 = [(P-Q)^d]^2P$.

Proof. (1) See Theorem 4.7, item (2) and item (3).

(2) Note that $(P-PQ)^2 = P-PQ$ and $(P-PQ)^d = P-PQ$. By Theorem 4.7, item (4), we get $(I-PQ)^dP = P$. By Lemma 1.2, we get

$$
\begin{aligned}
(P+Q-PQ)^d &= [P(I-Q)+Q]^d \\
&= (I-Q)(P-PQ) + (I-Q)\left[\sum_{n=1}^{\infty} Q^n(P-PQ)^{n+1}\right] \\
&\quad + Q[I-(P-PQ)] + \left[\sum_{n=1}^{\infty} Q^{n+1}(P-PQ)^n\right][I-(P-PQ)] \\
&= (I-Q)(P-PQ) + Q[I-(P-PQ)] \\
&= P+Q-PQ-2QP+2QPQ.
\end{aligned}
$$

Finally, by Theorem 4.3, we obtain

$$
\begin{aligned}
(P-Q)^d &= (I-PQ)^d(P-PQ) + (P+Q-PQ)^d(PQ-Q) \\
&= P(I-Q) + (P+Q-PQ-2QP+2QPQ)(PQ-Q) \\
&= P-PQ+PQ-Q-QPQ \\
&= P-Q-QPQ.
\end{aligned}
$$

By Theorem 4.2, item (1), we obtain

$$
\begin{aligned}
(P+Q)^d &= (P-Q)^d(P+Q)(P-Q)^d \\
&= (P-Q-QPQ)(P+Q)(P-Q-QPQ) \\
&= P+Q-2PQ-2QP+3QPQ.
\end{aligned}
$$

(3) See Theorem 4.4 and Theorem 4.5.
(4) See Theorem 4.7, item (4). $\square$

Proceeding similarly, we can derive more results as follows.

Corollary 5.2. *Let P and Q be two idempotents. If $PQP = P$, then*

(1) $(P - PQ)^d = 0, (P - QP)^d = 0, (PQ - QP)^d = 0.$
(2) ([14, Theorem 2.3])

$$(P - Q)^D = Q(P - I)Q,$$
$$(P + Q)^D = \frac{1}{8}(P + Q)^2 + \frac{7}{8}Q(I - P)Q,$$

(3) $(I - PQ)^d P = P(I - QP)^d = 0,$
(4) $(PQ)^d = PQ, P = [(I - P - Q)^d]^2 P.$

Corollary 5.3. *Let P and Q be two idempotents. If $PQP = PQ$, then*

(1) $(P - PQ)^d = P - PQ, (I - PQ)^d = I - PQ,$
(2) ([14, Theorem 2.6])

$$(P + Q)^D = P + Q - 2QP - \frac{3}{4}PQ + \frac{5}{4}QPQ,$$
$$(P - Q)^D = P - Q - PQ + QPQ,$$

(3) $(PQ)^d = [(I - P - Q)^d]^4 PQ, (PQ - QP)^d = 0,$
(4) $(P - QP)^d = P - PQ - QP + QPQ.$

Corollary 5.4. *Let P and Q be two idempotents. If $PQP = QP$, then*

(1) $(P - QP)^d = P - QP, (I - QP)^d = I - QP,$
(2) ([14, Theorem 2.7])

$$(P + Q)^D = P + Q - 2PQ - \frac{3}{4}QP + \frac{5}{4}QPQ,$$
$$(P - Q)^D = P - Q - QP + QPQ,$$

(3) $(PQ)^d = [(I - P - Q)^d]^4 PQ, (PQ - QP)^d = 0,$
(4) $(P - PQ)^d = P - PQ - QP + QPQ.$

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